Path integral for relativistic oscillators: model of the Klein–Gordon particle in AdS space

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Abstract. Explicit path integration is carried out for the Green's functions of special relativistic harmonic oscillators in $(1 + 1)$- and $(3 + 1)$-dimensional Minkowski space-time modeled by a Klein–Gordon particle in the universal covering space-time of the anti-de Sitter static space-time. The energy spectrum together with the normalized wave functions are obtained. In the non-relativistic limit, the bound states of the one- and three-dimensional ordinary oscillators are regained.

1 Introduction

Relativistic problems that can be solved exactly by the use of the path integral approach are very limited especially for two reasons.
(1) For a relativistic particle with spin, the propagator cannot be described by a simple path integral based on any reasonable action. The fact that the spin has no classical origin makes it difficult to propose for it continuous paths [1].
(2) If the particles interact with each other or with an external potential, they can produce quantum effects which cannot be described by path fluctuations alone. These effects can be handled by perturbation theory in the framework of quantum field theory [2].

However, in recent years, there have been a few successful examples where the difficulty which concerns the spin has been handled. The Dirac propagator for a free particle [3] has been derived in the framework of a model where the spin is classically described by internal variables. The path integral treatments of the Dirac–Coulomb problem [4] and a Dirac electron in a one-dimensional Coulomb potential on the half-line and in the presence of an external superstrong magnetic field [5] have been obtained via the Biedenharn transformation [6]. The electron in the presence of a constant magnetic field [7] and the problem of charged particles in interaction with an electromagnetic plane wave alone [8] or plus a parallel magnetic field [9] have been studied by introducing a fifth parameter in order to bring the problem into a non-relativistic form. The relativistic spinless Coulomb system [10] and the Klein–Gordon particle in vector plus scalar Hulthén-type potentials [11] have also been solved by path integration.

Recently, from various points of view ([12–16] and references therein), there has been renewed interest for the relativistic harmonic oscillators because of a crucial point. Indeed, a simple replacement of the coordinates and generalized momenta in the corresponding classical Hamiltonian by their quantum mechanical counterparts is, in general, not correct, since the ambiguity resulting from ordering the operators must be resolved. To parameterize the operator ordering ambiguity of the position and the momentum operators, we show that it is necessary to introduce two parameters α and β which cannot be freely chosen. The problem of the quantum relativistic oscillators represented by quantum free relativistic particles on the universal covering space-time of the anti-de Sitter static space-time (CAdS) is a model characterized by a constraint on these parameters. This model is called one of "special quantum relativistic oscillators" in the sense that α and β are chosen to adjust the non-relativistic limit and to preserve the reality of the energy spectrum of the physical system.

To our knowledge, there is no path integral discussion for the quantum relativistic harmonic oscillators. The purpose of the present paper is to fill this gap. The treatment will be restricted to spinless systems.

Our study is organized in the following way: in Sect. 2, we construct the path integral associated with the $(1+1)$-dimensional special relativistic harmonic oscillator. The Green's function is derived in closed form, from which we obtain the energy spectrum and the normalized wave functions. In Sect. 3, we extend the discussion to the $(3+1)$-dimensional case. The radial Green's function is also given in closed form. The energy levels and the normalized wave functions are then deduced. Section 4 will contain our conclusion.

2 The $(1 + 1)$-dimensional special relativistic oscillator

The relativistic harmonic oscillator interaction in $(1 + 1)$ Minkowski space-time is equivalent to a free relativistic particle in the universal covering space-time of the anti-de Sitter space-time (CAdS). For a static form of the anti-de Sitter space-time metric, the line element is given by

$$\mathrm{d}s^2 = \Lambda(x)c^2\mathrm{d}t^2 - \frac{1}{\Lambda(x)}\mathrm{d}x^2, \tag{1}$$

where

$$\Lambda(x) = 1 + \frac{\omega^2}{c^2}x^2. \tag{2}$$

Classical mechanics is described in this space by the classical Lagrangian and Hamiltonian, respectively:

$$L = -Mc\sqrt{1 - \frac{1}{\Lambda(x)}\frac{v^2}{c^2} + \frac{\omega^2}{c^2}x^2}, \tag{3}$$

$$H^2 = M^2c^4 + p^2c^2 + M^2\omega^2c^2x^2 + 2\omega^2x^2p^2 + \frac{\omega^4}{c^2}x^4p^2. \tag{4}$$

If we proceed by adopting the substitutions $H \to \widehat{P}_0 = \mathrm{i}\hbar\frac{\partial}{\partial t}$, $p \to \widehat{P}_x = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}$, $x \to \widehat{x}$, there is an ambiguity which results from ordering the operators in the quantum mechanical counterpart of (4). Since there exist different ways to put the terms x^4p^2 and x^2p^2 into symmetrically ordered forms, we can construct a number of Hermitian mechanical quantum counterparts of (4). In order to avoid this ambiguity, we write all the Hermitian forms for each term as a linear combination. Whence, after calculation of all the commutators, we find the following replacements:

$$\begin{cases} x^4p^2 \to -\hbar^2\left(x^4\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 4x^3\frac{\mathrm{d}}{\mathrm{d}x} + \alpha x^2\right), \\ x^2p^2 \to -\hbar^2\left(x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}}{\mathrm{d}x} + \beta\right), \end{cases} \tag{5}$$

where the parameters α and β occur; they will be fixed below.

The Green's function $G(x'', x')$ that we consider obeys the Klein–Gordon equation

$$\left\{\Box + \kappa\Lambda(x) + (1 - \alpha + 2\beta)\frac{\omega^4}{c^4}x^2 + \frac{\omega^2}{c^2}\right\}G(x'', x')$$
$$= -\frac{1}{\hbar^2c^2}\delta(x'' - x'), \tag{6}$$

where

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Lambda(x)\frac{\partial^2}{\partial x^2}\Lambda(x), \tag{7}$$

and

$$\kappa = \left(\frac{Mc}{\hbar}\right)^2 + (1 - 2\beta)\frac{\omega^2}{c^2}. \tag{8}$$

Note that choosing to work with the symmetrically ordered form (7) of the d'Alembertian operator, the quantization of the original problem will not be modified.

By using Schwinger's integral representation [17], the solution of the differential equation (6) can be written as follows:

$$G(x'', x') = \frac{1}{2\mathrm{i}\hbar c^2}\int_0^\infty \mathrm{d}\lambda\,\langle x'', t''|\exp\left[\frac{\mathrm{i}}{\hbar}\widehat{H}\lambda\right]|x', t'\rangle, \tag{9}$$

where the integrand $\langle x'', t''|\exp\left[\frac{\mathrm{i}}{\hbar}\widehat{H}\lambda\right]|x', t'\rangle$ is similar to the propagator of a quantum system evolving in λ time from (x', t') to (x'', t'') with the effective Hamiltonian,

$$\widehat{H} = \frac{1}{2}\left[-\Lambda(x)\widehat{P}_x^2\Lambda(x) + \frac{\widehat{P}_0^2}{c^2} - \hbar^2\kappa\Lambda(x)\right.$$
$$\left. - \hbar^2(1 - \alpha + 2\beta)\frac{\omega^4}{c^4}x^2 - \frac{\hbar^2\omega^2}{c^2}\right]. \tag{10}$$

The integrand in (9) may be written as the path integral [8,9,11,18,19]

$$P(x'', t'', x', t'; \lambda) = \langle x'', t''|\exp\left[\frac{\mathrm{i}}{\hbar}\widehat{H}\lambda\right]|x', t'\rangle \tag{11}$$

$$= \lim_{N\to\infty}\int\prod_{n=1}^N \mathrm{d}x_n\mathrm{d}t_n\prod_{n=1}^{N+1}\frac{\mathrm{d}(P_x)_n}{2\pi\hbar}\frac{\mathrm{d}(P_0)_n}{2\pi\hbar}\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{n=1}^{N+1}A_1^\varepsilon\right\},$$

with the short-time action

$$A_1^\varepsilon = (P_0)_n\triangle t_n - (P_x)_n\triangle x_n$$
$$+ \frac{\varepsilon}{2}\left(\frac{(P_0)_n^2}{c^2} - \Lambda(x_n)\Lambda(x_{n-1})(P_x)_n^2\right.$$
$$\left. - \hbar^2\frac{\omega^2}{c^2} - \hbar^2\kappa\Lambda(x_n) - \hbar^2(1 - \alpha + 2\beta)\frac{\omega^4}{c^4}x_n^2\right), \tag{12}$$

where

$$\varepsilon = \frac{\lambda}{N+1} = s_n - s_{n-1}, \tag{13}$$

and $s \in [0, \lambda]$ is a new time-like variable.

Let us first notice that the integrations on the variables t_n give N Dirac distributions $\delta((P_0)_n - (P_0)_{n+1})$. Thereafter the integrations on $(P_0)_n$ give $(P_0)_1 = (P_0)_2 = \dots = (P_0)_{N+1} = E$. The propagator (11) then becomes

$$P(x'', t'', x', t'; \lambda) \tag{14}$$
$$= \int_{-\infty}^{+\infty}\frac{\mathrm{d}E}{2\pi\hbar}\exp\left[-\frac{\mathrm{i}}{\hbar}E(t'' - t')\right]P_E(x'', x'; \lambda),$$

where the kernel $P_E(x'', x'; \lambda)$ is given by

$$P_E(x'', x'; \lambda) \tag{15}$$
$$= \lim_{N\to\infty}\int\prod_{n=1}^N \mathrm{d}x_n\prod_{n=1}^{N+1}\frac{\mathrm{d}(P_x)_n}{2\pi\hbar}\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{n=1}^{N+1}A_2^\varepsilon\right],$$

with the short-time action

$$A_2^\varepsilon = -(P_x)_n\triangle x_n + \frac{\varepsilon}{2}\left[-\Lambda(x_n)\Lambda(x_{n-1})(P_x)_n^2\right.$$
$$\left. + \frac{\hbar^2\omega^2}{c^2}\left(\frac{E^2}{\hbar^2\omega^2} - 1\right)\right.$$
$$\left. - \hbar^2\kappa\Lambda(x_n) - \hbar^2(1 - \alpha + 2\beta)\frac{\omega^4}{c^4}x_n^2\right]. \tag{16}$$

Note that (14) is invariant under the change $E \to -E$.

Then, by integrating with respect to the variables $(P_x)_n$, we get

$$P_E(x'', x'; \lambda) = \frac{1}{\sqrt{\Lambda(x')\Lambda(x'')}} \lim_{N\to\infty} \prod_{n=1}^{N+1} \left[\frac{1}{2i\pi\hbar\varepsilon} \right]^{1/2}$$
$$\times \prod_{n=1}^{N} \left[\int \frac{dx_n}{\Lambda(x_n)} \right] \exp\left[\frac{i}{\hbar} \sum_{n=1}^{N+1} A_3^\varepsilon \right], \quad (17)$$

with the short-time action in configuration space

$$A_3^\varepsilon = \frac{\triangle x_n^2}{2\varepsilon\Lambda(x_n)\Lambda(x_{n-1})} + \frac{\varepsilon}{2}\hbar^2 \left[\left(\frac{E^2}{\hbar^2\omega^2} - 1 \right) \frac{\omega^2}{c^2} \right.$$
$$\left. - \kappa\Lambda(x_n) - (1 - \alpha + 2\beta)\frac{\omega^4}{c^4}x_n^2 \right]. \quad (18)$$

Substituting (14) into (9), we can rewrite (9) in the form

$$G(x'', x') = \int_{-\infty}^{+\infty} \frac{dE}{2\pi\hbar} \exp\left[-\frac{i}{\hbar}E(t'' - t') \right] G_E(x'', x'), \quad (19)$$

with

$$G_E(x'', x') = \frac{1}{2i\hbar c^2} \int_0^\infty d\lambda P_E(x'', x'; \lambda). \quad (20)$$

If we now introduce a new variable u_n together with a rescaling of time [20] from ε to σ_n given by

$$\begin{cases} x_n = \frac{c}{\omega}\sinh u_n, \\ \varepsilon = \sigma_n \frac{c^2}{\omega^2}\frac{1}{\cosh u_n \cosh u_{n-1}}, \end{cases} \quad (21)$$

and incorporate the constraint

$$\lambda = \frac{c^2}{\omega^2} \int_0^S \frac{ds}{\cosh^2 u}, \quad (22)$$

by using the identity

$$\frac{c^2/\omega^2}{\cosh u'' \cosh u'} \int_0^\infty dS\delta\left(\lambda - \frac{c^2}{\omega^2}\int_0^S \frac{ds}{\cosh^2 u} \right) = 1, \quad (23)$$

the path integral (20) can be written as

$$G_E(x'', x') = \frac{1}{2i\hbar\omega c \left(\cosh u'' \cosh u'\right)^{3/2}}$$
$$\times \int_0^\infty dSP(u'', u'; S), \quad (24)$$

where

$$P(u'', u'; S)$$
$$= \lim_{N\to\infty} \int \prod_{n=1}^{N+1} \left[\frac{1}{2i\pi\hbar\sigma_n} \right]^{1/2} \prod_{n=1}^{N} du_n \exp\left\{ \frac{i}{\hbar} \sum_{n=1}^{N+1} \left[\frac{\triangle u_n^2}{2\sigma_n} \right.\right.$$
$$+ \frac{\triangle u_n^4}{8\sigma_n}\left(\frac{1}{3} - \frac{1}{\cosh^2 \widetilde{u}_n} \right) - \frac{\hbar^2}{2}\left(\frac{c^2}{\omega^2}\kappa - \frac{E^2/\hbar^2\omega^2 - 1}{\cosh^2 \widetilde{u}_n} \right.$$
$$\left.\left.\left. + (1 - \alpha + 2\beta)\tanh^2 \widetilde{u}_n \right)\sigma_n \right] \right\}. \quad (25)$$

Here, we have used the usual abbreviations $\triangle u_n = u_n - u_{n-1}$, $\widetilde{u}_n = (u_n + u_{n-1})/2$, $u' = u(0)$ and $u'' = u(S)$. Note that the term in $(\triangle u_n)^4$ contributes significantly to the path integral. It can be estimated by using the formula [21]

$$\int_{-\infty}^{+\infty} \exp(-\alpha_1 x^2 + \alpha_2 x^4)dx$$
$$= \int_{-\infty}^{+\infty} \exp\left(-\alpha_1 x^2 + \frac{3\alpha_2}{4\alpha_1^2} \right) dx, \quad (26)$$

valid for $|\alpha_1|$ large and $\mathrm{Re}(\alpha_1) > 0$. This leads to

$$P(u'', u'; S) \quad (27)$$
$$= \int Du(s) \exp\left\{ \frac{i}{\hbar} \int_0^S \left[\frac{\dot{u}^2}{2} - \frac{\hbar^2}{2}\left(\frac{c^2}{\omega^2}\kappa + \frac{1}{4} \right) \right.\right.$$
$$\left.\left. + \frac{\hbar^2}{2}\frac{E^2/\hbar^2\omega^2 - 1/4}{\cosh^2 u} - \frac{\hbar^2}{2}(1 - \alpha + 2\beta)\tanh^2 u \right]ds \right\}.$$

By noting that $\tanh^2 u = 1 - (1/\cosh^2 u)$, this last path integral is identical in form with that of the symmetric Rosen–Morse potential [22] which has been studied recently [23–27], but in order to obtain the equivalent to the Klein–Gordon equation in the AdS space-time we impose a restriction on the parameters α and β defined by the following two equations:

$$1 - \alpha + 2\beta = 0, \quad (28)$$
$$(1 - 2\beta)\frac{\omega^2}{c^2} = \xi R, \quad (29)$$

where $R = -2\omega^2/c^2$ is the scalar curvature and ξ is a numerical factor. From this it follows that

$$\alpha = 2\xi + 2 \quad \text{and} \quad \beta = \xi + \frac{1}{2}. \quad (30)$$

In this case, the propagator (27) reduces to

$$P(u'', u'; S)$$
$$= \int Du(s) \exp\left\{ \frac{i}{\hbar} \int_0^S \left[\frac{\dot{u}^2}{2} - \frac{\hbar^2}{2}\left(\left(\frac{Mc^2}{\hbar\omega} \right)^2 - 2\xi + \frac{1}{4} \right) \right.\right.$$
$$\left.\left. + \frac{\hbar^2}{2}\frac{E^2/\hbar^2\omega^2 - 1/4}{\cosh^2 u} \right]ds \right\}, \quad (31)$$

which is likewise the propagator relative to a symmetric Rosen–Morse potential. The Green's function associated

with this potential has been evaluated through various path integration techniques [23–27]. The result is

$$
G(u'', u'; E) = \int_0^{+\infty} \mathrm{d}S P(u'', u'; S)
$$
$$
= -\frac{\mathrm{i}}{\hbar} \Gamma(\gamma - l_E) \Gamma(1 + l_E + \gamma)
$$
$$
\times P_{l_E}^{-\gamma}(\tanh u'') P_{l_E}^{-\gamma}(-\tanh u'), \quad (32)
$$

where $P_{l_E}^{-\gamma}(\tanh u)$ is the associated Legendre function with

$$
l_E = -\frac{1}{2} + \frac{E}{\hbar\omega}, \qquad (33)
$$

and

$$
\gamma = \pm\frac{1}{2}\sqrt{1 + 4N^2 - 8\xi}, \quad N = \frac{Mc^2}{\hbar\omega}. \qquad (34)
$$

If we take into account (30), insert (32) into (24), and remember the first equation of the transformation (21), we obtain the Green's function for the one-dimensional special relativistic harmonic oscillator under consideration

$$
G_E(x'', x') = -\frac{\Gamma(\gamma - l_E)\Gamma(1 + l_E + \gamma)}{2\hbar^2\omega c}
$$
$$
\times \left[\left(1 + \frac{\omega^2}{c^2}x''^2\right)\left(1 + \frac{\omega^2}{c^2}x'^2\right)\right]^{-3/4} \qquad (35)
$$
$$
\times P_{l_E}^{-\gamma}\left(\frac{\frac{\omega}{c}x''}{\sqrt{1 + \frac{\omega^2}{c^2}x''^2}}\right) P_{l_E}^{-\gamma}\left(-\frac{\frac{\omega}{c}x'}{\sqrt{1 + \frac{\omega^2}{c^2}x'^2}}\right).
$$

The poles of the Green's function yield the discrete energy spectrum. These are just the poles of $\Gamma(\gamma - l_E)$ which occur when $\gamma - l_E = -n$ for $n = 0, 1, 2, ...$ They are given through the equations

$$
\frac{1}{2} - \frac{E}{\hbar\omega} \pm \frac{1}{2}\sqrt{1 + 4N^2 - 8\xi} = -n. \qquad (36)
$$

So, algebraically we obtain two distinct sets of energy levels according to the positive and negative signs of the parameter γ. But we have to check whether the corresponding wave functions, which will be expressed in terms of the Legendre functions of the first kind $P_{l_E}^{-\gamma}(y)$, satisfy the boundary conditions for

$$
y = \frac{\omega}{c}x/\sqrt{1 + \frac{\omega^2}{c^2}x^2} \to \pm 1.
$$

By inspecting their asymptotic behaviors [28]

$$
P_{l_E}^{-\gamma}(y) \underset{y\to 1}{\simeq} \frac{(1-y)^{\gamma/2}}{2^{\gamma/2}\Gamma(1+\gamma)};
$$
$$
\gamma \neq 0, -1, -2, -3, ..., \qquad (37)
$$

$$
P_{l_E}^{-\gamma}(y) \underset{y\to -1}{\simeq} \begin{cases} -\dfrac{\Gamma(-\gamma)}{2^{\gamma/2}\pi}\sin(l_E\pi)(1+y)^{\gamma/2} \\ \qquad \text{for} \quad \mathrm{Re}(\gamma) < 0, \\[2ex] \dfrac{2^{\gamma/2}\Gamma(\gamma)}{\Gamma(1+l_E+\gamma)\Gamma(\gamma - l_E)}(1+y)^{-\gamma/2} \\ \qquad \text{for} \quad \mathrm{Re}(\gamma) > 0, \end{cases} \qquad (38)
$$

we see that $P_{l_E}^{-\gamma}(y)$ diverges if $\mathrm{Re}(\gamma) < 0$. Therefore, we must choose the positive sign of γ, and hence the energy eigenvalues are

$$
E_n = \left(n + \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4N^2 - 8\xi}\right)\hbar\omega. \qquad (39)
$$

On the other hand, the reality of the parameter γ implies the following range of the numerical factor $\xi < (1/8) \times (1 + 4N^2)$.

In the limit $c \to \infty$, the energy spectrum approaches

$$
E_n^{\mathrm{NR}} + Mc^2 = \hbar\omega\left(n + \frac{1}{2}\right) + Mc^2. \qquad (40)
$$

The first term gives the energy levels in the non-relativistic case and the second term is the rest energy of the harmonic oscillator.

The corresponding energy eigenfunctions can be found by approximation near the poles $\gamma - l_E \approx -n$:

$$
\Gamma(\gamma - l_E) \approx \frac{(-1)^n}{n!}\frac{1}{\gamma - l_E + n}
$$
$$
= \frac{(-1)^{n+1}}{n!}\frac{2\left(n + \gamma + \frac{1}{2}\right)\hbar^2\omega^2}{E^2 - E_n^2}. \qquad (41)
$$

Using this behavior and the known property of the symmetry of the associated Legendre functions under spatial reflection, $x \to -x$, we get the contribution of the bound states to the spectral representation of the Green's function as

$$
G_E(x'', x') = \sum_{n=0}^{\infty}\frac{\omega}{c}\frac{n + \gamma + \frac{1}{2}}{n!\,(E^2 - E_n^2)}\Gamma(2\gamma + n + 1)
$$
$$
\times \left(1 + \frac{\omega^2}{c^2}x''^2\right)^{-3/4}\left(1 + \frac{\omega^2}{c^2}x'^2\right)^{-3/4}
$$
$$
\times P_{n+\gamma}^{-\gamma}\left(\frac{\frac{\omega}{c}x''}{\sqrt{1 + \frac{\omega^2}{c^2}x''^2}}\right)
$$
$$
\times P_{n+\gamma}^{-\gamma}\left(\frac{\frac{\omega}{c}x'}{\sqrt{1 + \frac{\omega^2}{c^2}x'^2}}\right)
$$
$$
= \sum_{n=0}^{\infty}\frac{\Psi_n^\gamma(x'')\Psi_n^{\gamma*}(x')}{E^2 - E_n^2}. \qquad (42)
$$

The properly normalized wave functions are thus

$$\Psi_n^\gamma(x) = \left[\frac{\omega}{c}\frac{n+\gamma+\frac{1}{2}}{n!}\Gamma(2\gamma+n+1)\right]^{1/2} \tag{43}$$

$$\times\left(1+\frac{\omega^2}{c^2}x^2\right)^{-3/4}P_{n+\gamma}^{-\gamma}\left(\frac{\frac{\omega}{c}x}{\sqrt{1+\frac{\omega^2}{c^2}x^2}}\right).$$

Taking into account the relation between the Gegenbauer polynomials and the associated Legendre functions (see formula (8.936) p. 1031 in [29])

$$C_n^\lambda(t) = \frac{\Gamma(2\lambda+n)\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(2\lambda)\Gamma(n+1)}$$

$$\times\left[\frac{1}{4}\left(t^2-1\right)\right]^{1/4-\lambda/2}P_{\lambda+n-1/2}^{1/2-\lambda}(t), \tag{44}$$

and using the doubling formula (see (8.335.1), p. 938 in [29])

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right), \tag{45}$$

we can also express (43) in the form

$$\Psi_n^\gamma(x) = \left[\frac{\omega}{c}\frac{\left(\gamma+n+\frac{1}{2}\right)n!}{\Gamma(2\gamma+n+1)}\right]^{1/2}$$

$$\times(2i)^\gamma\Gamma\left(\gamma+\frac{1}{2}\right)\left(1+\frac{\omega^2}{c^2}x^2\right)^{-(1/2)(\gamma+(3/2))}$$

$$\times C_n^{\gamma+1/2}\left(\frac{\frac{\omega}{c}x}{\sqrt{1+\frac{\omega^2}{c^2}x^2}}\right). \tag{46}$$

In the limit $c\to\infty$, $\gamma\to N=(Mc^2)/(\hbar\omega)$ and with the help of the formula (see (8.328.1), p. 937 in [29])

$$\lim_{z\to\infty}\frac{\Gamma(z+a)}{\Gamma(z)}e^{-a\ln z} = 1, \tag{47}$$

we see that

$$\lim_{c\to\infty}\gamma^{n/2}\left[\frac{\omega}{c}\frac{\left(\gamma+n+\frac{1}{2}\right)n!}{\Gamma(2\gamma+n+1)}\right]^{1/2}(2)^\gamma\Gamma\left(\gamma+\frac{1}{2}\right)$$

$$=\lim_{c\to\infty}\left[\frac{\omega}{c\sqrt{\pi}}\frac{n!}{2^n}\right]^{1/2}\left[\frac{\Gamma\left(\gamma+\frac{1}{2}\right)}{\Gamma(\gamma)}\right]^{1/2}$$

$$=\left(\frac{M\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{n!}{2^n}}. \tag{48}$$

By the use of the limit relation (see (8.936.5), p. 1031 in [29])

$$\lim_{\lambda\to\infty}\lambda^{-n/2}C_n^{\lambda/2}\left(t\sqrt{\frac{2}{\lambda}}\right) = \frac{2^{-n/2}}{n!}H_n(t), \tag{49}$$

the wave functions of the harmonic oscillator in the non-relativistic approximation are naturally regained:

$$\lim_{c\to\infty}\Psi_n^\gamma(x) = \left(\frac{M\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}e^{-(M\omega)/(2\hbar)x^2}$$

$$\times H_n\left(\sqrt{\frac{M\omega}{\hbar}}x\right), \tag{50}$$

where $H_n\left((M\omega/\hbar)^{1/2}x\right)$ is the Hermite polynomial of nth order.

3 The $(3+1)$-dimensional special relativistic oscillator

The special relativistic harmonic oscillator in $(3+1)$ Minkowski space-time is simulated in the universal covering space-time (CAdS) of the anti-de Sitter space-time with a negative curvature $R = -12\omega^2/c^2$ and a static metric of the form

$$ds^2 = \Lambda(r)c^2 dt^2 - \frac{1}{\Lambda(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{51}$$

where

$$\Lambda(r) = 1 + \frac{\omega^2}{c^2}r^2 \tag{52}$$

is chosen in order to impose the non-relativistic limit.

The Lagrangian reads

$$L = -Mc\sqrt{\Lambda(r) - \frac{v^2}{c^2} + \frac{\omega^2}{c^4}\frac{(\vec{r}\,\vec{v})^2}{\Lambda(r)}}, \tag{53}$$

and the classical Hamiltonian is given by

$$H^2 = \Lambda(r)\left(M^2c^4 + p^2c^2 + \omega^2(\vec{r}\,\vec{p})^2\right). \tag{54}$$

As in the one-dimensional relativistic oscillator, to construct the quantum mechanical counterpart of (54), we must respect the ordering ambiguity of the position and momentum operators. Similarly to (5), we will be led to make the following substitutions:

$$\begin{cases} x_i^4 p_i^2 \to -\hbar^2\left(x_i^4\frac{\partial^2}{\partial x_i^2} + 4x_i^3\frac{\partial}{\partial x_i} + \alpha x_i^2\right), \\ x_i^2 p_i^2 \to -\hbar^2\left(x_i^2\frac{\partial^2}{\partial x_i^2} + 2x_i\frac{\partial}{\partial x_i} + \beta\right), \\ x_i^3 p_i \to -i\hbar\left(x_i^3\frac{\partial}{\partial x_i} + \frac{3}{2}x_i^2\right), \\ x_i p_i \to -i\hbar\left(x_i\frac{\partial}{\partial x_i} + \frac{1}{2}\right). \end{cases} \tag{55}$$

The Green's function $G(\vec{r}'', t'', \vec{r}', t')$ for the problem satisfies the Klein–Gordon equation

$$(\Box + U(r))\, G(\vec{r}'', t''; \vec{r}', t')$$
$$= -\frac{1}{\hbar^2 c^2} \delta\left(\vec{r}'' - \vec{r}'\right) \delta\left(t'' - t'\right), \qquad (56)$$

where

$$\Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Lambda(r) \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \Lambda(r) + \Lambda(r) \frac{\hat{l}^2}{\hbar^2 r^2}, \qquad (57)$$

($\hat{l}^2$ is the square of the orbital angular momentum operator), and $U(r)$ is the central potential

$$U(r) = \frac{\hbar^2 \omega^2}{c^2} \left[4\beta - \alpha - \frac{M^2 c^4}{\hbar^2 \omega^2} - 8 \right.$$
$$\left. + \left(\alpha + 2\beta + \frac{7}{2}\right) \Lambda(r) \right]. \qquad (58)$$

It follows that the Green's function $G(\vec{r}'', t'', \vec{r}', t')$ can be expanded into partial waves [30] in spherical polar coordinates

$$G(\vec{r}'', t'', \vec{r}', t') \qquad (59)$$
$$= \frac{1}{r'' r'} \sum_{l=0}^{\infty} G_l(r'', t'', r', t') Y_l^{m*}(\theta'', \phi'') Y_l^m(\theta', \phi'),$$

where the radial Green's function, expressed in the Schwinger integral representation [17], is

$$G_l(r'', t'', r', t') \qquad (60)$$
$$= \frac{1}{2i\hbar c^2} \int_0^\infty d\lambda \, \langle r'', t'' | \exp\left[\frac{i}{\hbar} \hat{H}_l \lambda\right] | r', t' \rangle.$$

The integrand in (60) is similar to the propagator of an harmonic oscillator which evolves in the time-like parameter λ with the effective Hamiltonian

$$\hat{H}_l = \frac{1}{2} \left[-\Lambda(r) \hat{P}_r^2 \Lambda(r) + \frac{\hat{P}_0^2}{c^2} \right.$$
$$\left. - \hbar^2 l(l+1) \frac{\Lambda(r)}{r^2} + U(r) \right]. \qquad (61)$$

To find the energy eigenvalues $E_{n_r, l}$ and the wave functions $\Psi_{n_r, l}(r) = r^{-1} \Phi_{n_r, l}(r)$, we may evaluate (60) by path integration. The effective Hamiltonian (60) involves a centrifugal barrier which possesses a singularity at $r = 0$, so that the discrete form of the expression (60) is not defined due to a path collapse. To obtain a tractable and stable path integral, we introduce an appropriate regulating function (following Kleinert [31]) and write (60) in the form

$$G_l(r'', t'', r', t') = \frac{1}{2i\hbar c^2} \int_0^\infty dS P_l(r'', t'', r', t'; S), \quad (62)$$

where the transformed path integral is given in the canonical form by

$$P_l(r'', t'', r', t'; S)$$
$$= f_R(r'') f_L(r') \langle r'', t'' | \exp\left[\frac{i}{\hbar} S f_L(r) \hat{H}_l f_R(r)\right] | r', t' \rangle$$
$$= f_R(r'') f_L(r') \int Dr(s) Dt(s) \int \frac{DP_r(s) DP_0(s)}{(2\pi\hbar)^2}$$
$$\times \exp\left\{\frac{i}{\hbar} \int_0^S ds \left[-P_r \, \dot{r} + P_0 \, \dot{t} + f_L(r) H_l f_R(r) \right] \right\}$$
$$= f_R(r'') f_L(r') \lim_{N\to\infty} \prod_{n=1}^{N} \left[\int dr_n dt_n \right]$$
$$\times \prod_{n=1}^{N+1} \left[\int \frac{d(P_r)_n d(P_0)_n}{(2\pi\hbar)^2} \right] \exp\left\{\frac{i}{\hbar} \sum_{n=1}^{N+1} A_1^{\varepsilon_s}\right\}, \quad (63)$$

with the short-time action

$$A_1^{\varepsilon_s} = -(P_r)_n \triangle r_n + (P_0)_n \triangle t_n$$
$$+ \frac{\varepsilon_s}{2} f_L(r_n) \left[-\Lambda(r_n) \Lambda(r_{n-1}) (P_r)_n^2 + \frac{(P_0)_n^2}{c^2} \right.$$
$$\left. - \hbar^2 l(l+1) \frac{\Lambda(r_n)}{r_n^2} + U(r_n) \right] f_R(r_{n-1}), \qquad (64)$$

and

$$\varepsilon_s = \frac{S}{N+1} = \triangle s_n = \frac{\triangle \tau_n}{f_L(r_n) f_R(r_{n-1})};$$
$$\triangle \tau_n = \varepsilon_\tau = \frac{\lambda}{N+1}. \qquad (65)$$

The regulating function is defined as [31]

$$f(r) = f_L(r) f_R(r) = f^{1-\lambda'}(r) f^{\lambda'}(r). \qquad (66)$$

As in the $(1+1)$-dimensional case, by doing successively the t_n and $(P_0)_n$ integrations we arrive at

$$P_l(r'', t'', r', t'; S) \qquad (67)$$
$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dE \exp\left[-\frac{i}{\hbar} E(t'' - t')\right] P_l(r'', r'; S),$$

where the invariant kernel $P_l(r'', r'; S)$ under the change $E \to -E$ is given by

$$P_l(r'', r'; S) = f_R(r'') f_L(r') \lim_{N\to\infty} \prod_{n=1}^{N} \left[\int dr_n \right] \qquad (68)$$
$$\times \prod_{n=1}^{N+1} \left[\int \frac{d(P_r)_n}{(2\pi\hbar)} \right] \exp\left\{\frac{i}{\hbar} \sum_{n=1}^{N+1} A_2^{\varepsilon_s}\right\},$$

with

$$A_2^{\varepsilon_s} = -(P_r)_n \triangle r_n$$
$$+ \frac{\varepsilon_s}{2} f_L(r_n) \left[-\Lambda(r_n) \Lambda(r_{n-1}) (P_r)_n^2 + \frac{E^2}{c^2} \right.$$
$$\left. - \hbar^2 l(l+1) \frac{\Lambda(r_n)}{r_n^2} + U(r_n) \right] f_R(r_{n-1}). \qquad (69)$$

Substituting (67) into (62), we observe that the t-dependent term does not contain the variable S. Therefore, we can rewrite the partial Green's function (62) in the form

$$G_l(r'', t'', r', t') \tag{70}$$
$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} \mathrm{d}E \exp\left[-\frac{\mathrm{i}}{\hbar} E(t'' - t')\right] G_l(r'', r'),$$

with

$$G_l(r'', r') = \frac{1}{2\mathrm{i}\hbar c^2} \int_0^\infty \mathrm{d}S P_l(r'', r'; S). \tag{71}$$

The path integration of the kernel $P_l(r'', r'; S)$ can be performed for any splitting parameter λ'. However, to simplify the calculation, we prefer to work with the mid-point prescription by taking $\lambda' = 1/2$. This can be justified by the fact that the final result is independent of this parameter. Then, by integrating with respect to the variables $(P_r)_n$, we find

$$P_l(r'', r'; S) \tag{72}$$
$$= \frac{[f(r')f(r'')]^{1/4}}{\sqrt{\Lambda(r')\Lambda(r'')}} \lim_{N\to\infty} \prod_{n=1}^{N+1} \sqrt{\frac{1}{2\mathrm{i}\pi\hbar\varepsilon_s}}$$
$$\times \prod_{n=1}^{N} \left[\int \frac{\mathrm{d}r_n}{\Lambda(r_n)\sqrt{f(r_n)}}\right] \exp\left\{\frac{\mathrm{i}}{\hbar} \sum_{n=1}^{N+1} A_3^{\varepsilon_s}\right\},$$

with the short-time action in configuration space

$$A_3^{\varepsilon_s} = \frac{\triangle r_n^2}{2\varepsilon_s \Lambda(r_n)\Lambda(r_{n-1})\sqrt{f(r_n)f(r_{n-1})}} \tag{73}$$
$$+ \frac{\varepsilon_s}{2} f(r_n) \left[\frac{E^2}{c^2} - \hbar^2 l(l+1)\frac{\Lambda(r_n)}{r_n^2} + U(r_n)\right].$$

We now use the following space transformation: $r \to u$, $r \in [0, \infty[$, $u \in]-\infty, \infty[$ defined by

$$r = \frac{c}{\omega} \mathrm{e}^u. \tag{74}$$

The appropriate regulating function is then defined by

$$f(r(u)) = \frac{c^2}{4\omega^2 \cosh^2 u}. \tag{75}$$

By taking into account all the quantum corrections arising, of course, from the transformations (74) and (75), the Green's function (71) can straightforwardly be written as follows:

$$G_l(r'', r') = \frac{1}{4\mathrm{i}\hbar\omega c \sqrt{\Lambda(r'')\Lambda(r')} \cosh u'' \cosh u'}$$
$$\times \int_0^\infty \mathrm{d}S P_l(u'', u'; S), \tag{76}$$

with

$$P_l(u'', u'; S)$$
$$= \int Du(s) \exp\left\{\frac{\mathrm{i}}{\hbar} \int_0^S \left[\frac{\dot{u}^2}{2} - \frac{\hbar^2}{4}\left(\nu^2 + k^2\right.\right.\right. \tag{77}$$

$$\left.\left.\left. + (\nu^2 - k^2)\tanh u + \frac{\hbar^2}{8} \frac{\frac{E^2}{\hbar^2\omega^2} + 4\beta - \alpha - 2}{\cosh^2 u}\right]\right\}\right\},$$

where $\nu^2 = N^2 - 2\beta - \alpha + (11/4)$, $k = l + (1/2)$ and $N = Mc^2/\hbar\omega$.

This kernel is formally identical with that of the general Rosen–Morse (or general modified Pöschl–Teller) potential studied recently [23–26]. The Green's function associated to this potential is

$$G(u'', u'; E_{RM'}) = \int_0^\infty \mathrm{d}S P_l(u'', u'; S). \tag{78}$$

As is shown by Kleinert [31], the Green's function of the general Rosen–Morse potential is related to the fixed-energy amplitude for the mass point subjected to an angular barrier near the surface of a sphere in $D = 4$ dimensions by

$$G(u'', u'; E_{RM'}) = \frac{1}{\sqrt{\sin\theta'' \sin\theta'}} G(\theta'', \theta'; E_{PT'}) \tag{79}$$
$$= -\frac{\mathrm{i}}{\hbar} \frac{\Gamma(M_1 - L_E)\Gamma(L_E + M_1 + 1)}{\Gamma(M_1 + M_2 + 1)\Gamma(M_1 - M_2 + 1)}$$
$$\times \left(\frac{1 + \tanh u'}{2}\right)^{(M_1 - M_2)/2} \left(\frac{1 - \tanh u'}{2}\right)^{(M_1 + M_2)/2}$$
$$\times \left(\frac{1 - \tanh u''}{2}\right)^{(M_1 + M_2)/2} \left(\frac{1 + \tanh u''}{2}\right)^{(M_1 - M_2)/2}$$
$$\times F\left(M_1 - L_E, L_E + M_1 + 1; M_1 - M_2 + 1; \frac{1 + \tanh u'}{2}\right)$$
$$\times F\left(M_1 - L_E, L_E + M_1 + 1; M_1 + M_2 + 1; \frac{1 - \tanh u''}{2}\right),$$

with $\tanh u = -\cos\theta, \theta \in (0, \pi), u \in]-\infty, +\infty[$ and $u'' > u'$. Here, the mass point is taken equal to unity. In addition, we set

$$\begin{cases} L_E = -\frac{1}{2} + \left(\frac{1}{16} + \frac{2E_{PT'}}{\hbar}\right)^{1/2}, \\ E_{PT'} = \frac{\hbar^2}{8}\left(\frac{E^2}{\hbar^2\omega^2} + 4\beta - \alpha - \frac{5}{4}\right), \end{cases} \tag{80}$$

and if we choose

$$\begin{cases} M_1 = \frac{1}{2}\left(\sqrt{N^2 - 2\beta - \alpha + \frac{11}{4}} + l + \frac{1}{2}\right), \\ M_2 = \frac{1}{2}\left(\sqrt{N^2 - 2\beta - \alpha + \frac{11}{4}} - l - \frac{1}{2}\right), \end{cases} \tag{81}$$

the boundary conditions for the wave functions appearing in (79) will be satisfied.

The equivalence between the relativistic harmonic oscillator interaction in $(3+1)$ Minkowski space-time and a free relativistic particle in CAdS is characterized by the following restriction on the parameters α and β:

$$\alpha = 8\xi, \qquad \beta = 2\xi + \frac{1}{4}. \tag{82}$$

Inserting (79) into (76), we get, for the radial Green's function, the closed form

$$G_l(r'', r') = -\frac{\Gamma(M_1 - L_E)\Gamma(L_E + M_1 + 1)}{4\hbar^2 \omega c \Gamma(M_1 + M_2 + 1)\Gamma(M_1 - M_2 + 1)}$$

$$\times (\Lambda(r'')\Lambda(r')\cosh u'' \cosh u')^{-1/2} \qquad (83)$$

$$\times \left(\frac{1 + \tanh u'}{2}\right)^{(M_1 - M_2)/2} \left(\frac{1 - \tanh u'}{2}\right)^{(M_1 + M_2)/2}$$

$$\times \left(\frac{1 - \tanh u''}{2}\right)^{(M_1 + M_2)/2} \left(\frac{1 + \tanh u''}{2}\right)^{(M_1 - M_2)/2}$$

$$\times F\left(M_1 - L_E, L_E + M_1 + 1; M_1 - M_2 + 1; \frac{1 + \tanh u'}{2}\right)$$

$$\times F\left(M_1 - L_E, L_E + M_1 + 1; M_1 + M_2 + 1; \frac{1 - \tanh u''}{2}\right).$$

The poles of (83) are all contained in the first Γ function in the numerator,

$$M_1 - L_E = -n_r. \qquad (84)$$

Converting this into energy by using (80), (81) and (84) yields

$$E_{n_r,l} = \beta_{n_r,l}\hbar\omega, \qquad (85)$$

with

$$\beta_{n_r,l} = 2n_r + l + \frac{1}{2}\sqrt{9 + 4N^2 - 48\xi} + \frac{3}{2}, \qquad (86)$$

where n_r is the radial quantum number and l the angular momentum. Here, the parameter ξ is subject to the condition $\xi < (1/48)(9 + 4N^2)$.

In the non-relativistic approximation

$$E_{n_r,l} \underset{c\to\infty}{\to} \left(2n_r + l + \frac{3}{2}\right)\hbar\omega + Mc^2, \qquad (87)$$

where the first term represents the well-known energy spectrum of the three-dimensional non-relativistic harmonic oscillator.

As in the one-dimensional case, the radial wave functions can be found by approximation near the poles $M_1 - L_E \approx -n_r$:

$$\Gamma(M_1 - L_E) \approx \frac{(-1)^{n_r}}{n_r!} \frac{1}{M_1 - L_E + n_r}$$

$$= \frac{(-1)^{n_r+1}}{n_r!} \frac{4\hbar^2\omega^2\beta_{n_r,l}}{E^2 - E_{n_r,l}^2}. \qquad (88)$$

Using this behavior and taking into consideration the Gauss transformation formula (see (9.131.2), p. 1043 in [29])

$$F(a, b, c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$\times F(a, b, a+b-c+1; 1-z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \qquad (89)$$

$$\times (1-z)^{c-a-b}F(c-a, c-b, c-a-b+1; 1-z),$$

knowing that the second term of this latter is null because the Euler function $\Gamma(a)$ is infinite ($a = -n_r \leq 0$), we can write (83) as

$$G_l(r'', r') = \sum_{n_r=0}^{\infty} \frac{\Phi_{n_r,l}(r'')\Phi_{n_r,l}^*(r')}{E^2 - E_{n_r,l}^2}, \qquad (90)$$

where

$$\Phi_{n_r,l}(r) = \left[\frac{2\beta_{n_r,l}\Gamma(\beta_{n_r,l} - n_r)\Gamma\left(n_r + l + \frac{3}{2}\right)}{n_r!\Gamma\left(\beta_{n_r,l} - n_r - \frac{1}{2}\right)}\right]^{1/2}$$

$$\times \frac{1}{\Gamma\left(l + \frac{3}{2}\right)} \left(\frac{\omega}{c}\right)^{l+3/2} r^l \left(1 + \frac{\omega^2}{c^2}r^2\right)^{n_r - \beta_{n_r,l}/2}$$

$$\times F\left(-n_r, \beta_{n_r,l} - n_r; l + \frac{3}{2}; \frac{\frac{\omega^2}{c^2}r^2}{1 + \frac{\omega^2}{c^2}r^2}\right) \qquad (91)$$

are the radial wave functions.

By substituting (see (9.131.1), p. 1043 in [29])

$$F(\alpha, \beta; \gamma; z) = (1-z)^{-\alpha}F\left(\alpha, \gamma - \beta; \gamma; \frac{z}{z-1}\right) \qquad (92)$$

into (91) and using the connecting formula (see (8.962.1), p. 1036 in [29])

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \qquad (93)$$

$$\times F\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right),$$

we can also express (91) in the form

$$\Phi_{n_r,l}(r) = \left[\frac{2\beta_{n_r,l}n_r!\Gamma(\beta_{n_r,l} - n_r)}{\Gamma(n_r + \gamma + 1)\Gamma\left(n_r + l + \frac{3}{2}\right)}\right]^{1/2}$$

$$\times \left(\frac{\omega}{c}\right)^{l+3/2} r^l \left(1 + \frac{\omega^2}{c^2}r^2\right)^{-(1/2)\beta_{n_r,l}}$$

$$\times P_{n_r}^{(l+1/2, -\beta_{n_r,l})}\left(1 + 2\frac{\omega^2}{c^2}r^2\right). \qquad (94)$$

By using the following limiting relations:

$$\begin{cases} \lim_{\gamma\to\infty}\Gamma(n+\gamma+1) = \lim_{\gamma\to\infty}\gamma^{n+1}\Gamma(\gamma), \\ \lim_{\gamma\to\infty}\left(1 + \frac{\omega^2}{c^2}r^2\right)^{-(1/2)(l+\gamma+3/2)} = e^{-(M\omega)/(2\hbar)r^2}, \\ \lim_{\gamma\to\infty}F\left(\alpha, \gamma; \nu; \frac{z}{\gamma}\right) = F(\alpha, \nu; z), \end{cases} \qquad (95)$$

we obtain the well-known radial wave functions of the non-relativistic harmonic oscillator

$$
\begin{aligned}
\Phi_{n_r,l}(r) = {} & \left[\frac{2\Gamma\left(n_r + l + \dfrac{3}{2}\right)}{n_r!} \right]^{1/2} \left(\frac{M\omega}{\hbar} \right)^{(1/2)(l+3/2)} \\
& \times \frac{r^l}{\Gamma\left(l + \dfrac{3}{2}\right)} \\
& \times \exp\left[-\frac{M\omega}{2\hbar} r^2 \right] F\left(-n_r, l + \frac{3}{2}; \frac{M\omega}{\hbar} r^2 \right).
\end{aligned}
\tag{96}
$$

4 Conclusion

In this paper we have dealt with special relativistic harmonic oscillators in $(1+1)$- and $(3+1)$-dimensional Minkowski space-time modeled by a free relativistic particle in the universal covering space-time of the anti-de Sitter space-time. The explicit path integral solution, as presented above, provides a valuable alternative way to the one obtained through the Klein–Gordon equation. After formulating the problem in terms of symmetric and general Rosen–Morse potentials for the one- and three-dimensional relativistic oscillators, respectively, and by imposing a restriction on the parameters α and β in such a way that the system under consideration is equivalent to a free relativistic particle in CAdS, the Green's functions are obtained in a closed form. The energy spectrum and the properly normalized wave functions are extracted from the poles and the residues at the poles of the Green's function, respectively. In the flat-space limit $(R \to 0)$, that is to say in the non-relativistic approximation $(c \to \infty)$, the usual harmonic oscillator spectrum and the corresponding normalized wave functions are regained.

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